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Analytic Determination of the Tension Capable Workspace of Cable Actuated Haptic Interfaces

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Abstract

Cable robots or wire driven robots possess many advantages that makes them very well suited to be used as haptic interfaces. They exhibit very low inertia and very low friction because of their very light mechanical structure. The use of cables, however, leads to an under-constrained system which shows complex properties. The main drawback is the difficulty to estimate the useful workspace, and the maximum tension in the cables, as the relation between the maximum force at the tip of the interface and the maximum tension in the cables can not be easily established. In this paper we present a method to calculate these tensions in a given workspace and to estimate what we have called "the tension capable workspace" for a 3 cables 2 d.o.f (degree of freedom) planar haptic interface and for a 4 cables 2 d.o.f. one.

Keywords: Haptic interface, cable robot, redundant parallel robot,"Tension capable workspace".

1. Introduction

Cable or wire driven robots present many advantages:

- a very low structure weight, thus a very low inertia
- a high rigidity
- a potentially very large workspace as there is no internal mechanical structure to limit the movement of the end effector

These advantages are the main qualities expected for an haptic interface and some interfaces of this kind have already been built [2]. An example is the "Spidar" which is a 3 d.o.f (degree of freedom), 4 cables haptic interface, and its derivated configurations: the Spidar-G, the Spidar-8, and the wereable version called "hapticgear" [6]. The Laboratory of Automation and Robotics of the University of Bologna has also designed a wearable wire driven haptic interface called "VIDET" which is similar to the previous one [1].
However, the design of such haptic interfaces is difficult as it is not easy to establish a relation between the maximum force at the tip of the robot and the maximal tension in the cables. Some previous work has already been done to measure the performance of the wire actuated manipulators by the extension of manipulability ellipsoids or dexterity indices which are commonly used in the robotic theory. Krutz and Hayward [3] define dexterity measures based on the calculation of the maximum output force in the n+1 case, and Y. Shen and al.[9] introduce manipulability indices by calculating the volume of the ellipsoid defined by $F^TGG^TF \leq 1$. But the problem of finding the maximum tension for a given maximum output force in a given workspace, especially in the hyper redundant case (m $\geq$ n+2) remain still quite unexplored
In this paper we present a preliminary theoretical work done in order to answer this problem.
Firstly, we will remind the existing theory for the determination of the tensions in the cable and the calculation of the singularity free workspace. Secondly, we propose a new method to calculate the maximum tension in a cable for any output force, in any point of the workspace; and deduce what have called the "*Tension Capable Workspace*" defined as the workspace where the tension in the cables is less than a chosen maximum value.

2. General static model for cable robots

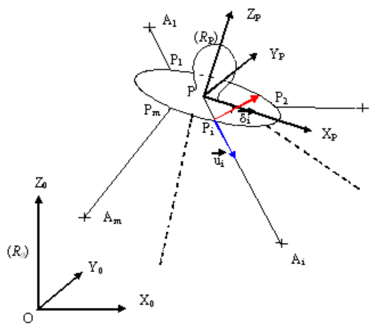

Figure 1: Diagram of a general cable driven haptic interface

Let's introduce the different parameters needed to define the static model of any kind of cable driven manipulators as schematized in fig 1:

- A_i: the connection point of the cable i to the motor m_i,
- P_i: the connection point of the cable i to the end effector,
- P: the center of the end effector,
- $l_i = ||\vec{A_iP_i}||$: the length of the cable i,
- $\vec{u_i}$: the unit vector along cable i,

- $\vec{\delta_i}$: the unit momentum vector around P_i, $\vec{\delta_i} = \vec{PP_i} \wedge \vec{u_i}$,
- $G = \begin{pmatrix} \vec{u}_1 \dots & \vec{u}_i \dots & \vec{u}_m \\ \vec{\delta}_1 \dots & \vec{\delta}_i \dots & \vec{\delta}_m \end{pmatrix} = [\mathbf{g}_1, \dots \mathbf{g}_n]$,
- $\theta_{ij} = (\vec{u_i}, \vec{u_j})$,
- t_i : the tension in the cable i,
- $\vec{F}_{out}$ and $\vec{M}_P^{out}$: the output force vector and momentum on the end effector expressed at P,
- n the number of d.o.f. of the interface.

i $\in$ [1,m] where m is the number of cables. Theoretical studies have shown that m must be superior or equal to the number of d.o.f +1 ($m \geq n+1$) [5].
All the coordinates of the different points are expressed in the base frame $R_0(\mathrm{O},\vec{i},\vec{j},\vec{k})$ which is assumed to be Galilean.

The equilibrium of the end effector gives:

$$\vec{F}_{out} = \sum_{i=1}^{m} t_i \vec{u_i} \text{ and } \vec{M}_P^{out} = \sum_{i=1}^{m} t_i \vec{\delta_i} \tag{1}$$

also expressed by:

$$\mathbf{F}_P = G\mathbf{T} \tag{2}$$

where : $\mathbf{F}_P = [\vec{F}_{out}, \vec{M}_P^{out}]^t$, $\mathbf{T} = [t_1, \dots, t_i, \dots, t_m]^t$

$\mathbf{T}$ is a m by 1 vector and G is a n by m matrix

2.1. Calculation of the tension in the cables for a given output force

2.1.1. First approach: Use of the Kernel

This approach has been widely used and explained ([4], [8]). It is based on the use of the pseudo inverse of G and the addition of the kernel to be able to have positive tensions in the cables.
As the matrix G is not square, G^{-1} can not be calculated. To compute the tension in the cables, we use the Moore-Penrose pseudo inverse $G^+ = G^t(GG^t)^{-1}$:

$$\mathbf{T} = G^+\mathbf{F}_P \tag{3}$$

The use of the pseudo inverse enables us to find a solution that minimizes the norm of $\mathbf{T}$. However the use of cables involves that each tension must be positive, otherwise the cables would slack. Unfortunately, the vector $\mathbf{T}$ previously found may have one or more of its component negative. In order to render all its components positive or equal to the minimum tension (t_{min}) without adding any extra force at the end effector, a solution is to add the kernel of G (K_G) multiplied by a constant α (which will be a scalar or a vector, depending of the dimension of the kernel). Thus we obtain :

$$\mathbf{T} = \underbrace{G^+\mathbf{F}_P}_{\vec{t_f}} + \alpha K_G \tag{4}$$

When the number of cables is equal to the number of d.o.f+1, K_G is a vector and the parameter α is a scalar easily determined by:

$$\alpha = \max_i\left(\frac{t_{min} - t_{f\,i}}{K_{G\,i}}\right) \tag{5}$$

2.1.2. Second approach: Calculation of the tension in the cable with the use of a reduced matrix

The first approach is wildly known and used, but it's quite time consuming because of the time taken for the calculation of G^+ and the elements of the kernel of G. In order to speed up the calculations, we have chosen to develop an other approach. For every point and any direction of the output force, outside singular configurations we can assume that we will be able to find at least a set S_k of n cables where the tension is strictly positive, $k \in [1, C_m^n]$ as there is $C_m^n = \frac{m!}{(m-n)!n!}$ possible choices of S_k.The tension in the $p = m - n$ cables left will be assumed to be t_{min}. This can be easily demonstrated for a 2 d.o.f., 3 cables planar translational interface.

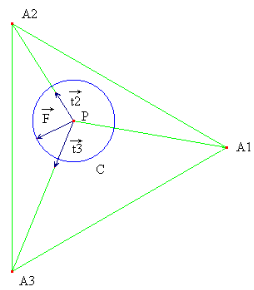

Figure 2: 2 d.o.f, 3 cables planar interface

For P inside $A_1A_2A_3$, and for any direction of $\vec{F_{out}}$ (see fig 2), $\vec{F_{out}}$ will always be "inside" two cables which angle is less tan π. It is then always possible to find a set of positive tensions along two cables which vectorial sum gives $\vec{F_{out}}$. The same demonstration can be done for the case of a 3 d.o.f., 4 cables translational interface considering that $\vec{F_{out}}$ will always be inside the pyramid formed by three of the four cables, for a carefully chosen workspace.
We have to solve a fully constrained system using n cables. To be able to find the right solution, we need to form all possible reduced systems and evaluate the corresponding values of the tensions in the cables . The solution which gives the lowest strictly positive tensions is the good solution.
Let's number the cables of the set S_k from c_1 to c_n[1]. We define f_k the function which gives the relation between the number of the cable c_i in S_k and the general number of the cable j.

$$\begin{array}{ccc} c_i & \xrightarrow{f_k} & j \\ [1,m] & & [1,m] \end{array}$$

The tension will be t_{min} in the p cables c_{n+1} to c_m,$\{c_1, \dots, c_m\} \in [1,m]^m$.

[1] By extension, we will also write $S_k = \{c_1, \dots, c_n\}$

We will have, for $det(G^{S_k}_{red}) \neq 0$:

$$[t_{c_{n+1}}, \ldots, t_{c_m}]^T = t_{min}$$
$$G^{S_k}_{red} = [\mathbf{g}_{c_1}, \ldots, \mathbf{g}_{c_n}]$$
$$\Leftrightarrow [t_{c_1}, \ldots, t_{c_n}]^T = {G^{S_k}_{red}}^{-1} \mathbf{F}^*_p$$

With $\mathbf{F}^*_P = \mathbf{F}_P - t_{min} \sum_{k=c_{n+1}}^{c_m} \vec{u_k}$

The tensions are then obtained by solving the previous Cramer system (6). Let's write:

$$A(F)^{S_k}_{c_r} = [\mathbf{g}_{c_1}, \ldots, \underbrace{\mathbf{F}^*_P}_{row\ c_r}, \ldots, g_{c_n}]$$

So:

$$\begin{cases} t^{S_k}_{c_r} = \dfrac{|A(F)^{S_k}_{c_r}|}{|G^{S_k}_{red}|} & \forall r \in [1, n] \\ t^{S_k}_{c_r} = t_{min} & \forall r \in [n+1, m] \end{cases} \tag{6}$$

This calculation has to be done for the C^n_m possible choices of S_k, and the solution adopted will be the one with the lowest strictly positive tensions. Let's S_g be this set. The tension t_i can be expressed by the following equation:

$$t_i = t^{S_g}_i = \begin{cases} \dfrac{|A(F)^{S_g}_i|}{|G^{S_g}_{red}|} & \forall i \in S_g \\ t_{min} & \forall i \notin S_g \end{cases} \tag{7}$$

2.1.3. Comparison of the two approaches

The two approaches have been compared with the example explained in 3.2.2 in terms of calculation's time. Because of redundancy, the first approach need the use of a quadratic algorithm to find λ with is a 2 by 1 vector. We find the following calculation's times (Tabular 1) for 15x16 points of simulation and 15 orientations of $\vec{F}_{out}$ on each point. This

First Approach	Our Approach
0,0131s	$5,69.10^{-4}$s

Table 1: Calculation's time for the two approaches

comparison between the two calculation's times is obviously linked to the example we have chosen. Further work has to been done to investigate the benefits of our approach in the general case.

2.2. Theoretic Workspace Analysis

The theoretical workspace (W_{th}) can be defined as the space where it is possible to find a positive tension in all cables to equilibrate the end effector for any force applied to it. This is possible :

- first, if the rank of G is equal to number of d.o.f of the robot [9],[7] (or when the rank of the kernel is equal to the difference between the number of cables and the number of d.o.f). For a under constrained point mass cable robot (m=n+1), the rank of the matrix G is less than the number of d.o.f when two cables are aligned or when three cables are in the same plane. The "singularity lines" are the lines crossing the points of connection of the cables.
- second, if the kernel is "cooperative". The kernel is said to be "cooperative"when all its components are strictly positive.

So the theoretical workspace will be the space between the previous boundaries where the kernel is cooperative. As demonstrated by Lafourcade for m=n+1 [4], W_{th} is the inside convex hyper-polyhedron formed by the cables connection points.

If we use the second approach, for m=n+1, this is also possible if we are able to find n cables among m . This is possible for any direction of the output force, if the matrices formed any set of n cables have their ranks equal to n. Any reduced matrix will lose its rank when one of the vector $\vec{u_i}$ is a linear combination on the n-1 other ones, i.e two cables are aligned or three cables are in the same plane. The inside polyhedron is then the place where the solution found by the use of the second approach will always be positive.

2.3. First conclusions on the design of cable haptic interfaces

To design such interface, the designer will have to carefully choose the architecture (i.e the position of the connection points of the cables A_i), in order to have the desired workspace (W_{pr}) included in the previous defined workspace W_{th}. The best choice would be to have the boundaries of W_{pr} as far as possible from the boundaries of W_{th} as the tensions in the cables increase when P gets closer to the boundaries of W_{th} . A possible way to increase W_{th} without increasing the size of the interface is to add more motors. We then modify the shape of W_{th} as shown in the following example (fig. 3)

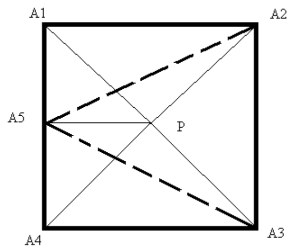

Figure 3: Two possible theoretical workspaces buy the use of 3 or 4 motors for a 2 d.o.f. planar interface

Using cables c_2, c_3, c_5 define the triangle A_2, A_3, A_5 as W_{th} , but using using cables c_1, c_2, c_3, c_4 define the square A_1, A_2, A_3, A_4 as W_{th}, witch is much larger.

3. Calculation of the maximum tension in a given workspace

3.1. Introduction

In a first step in the design process, a workspace is chosen with respect to ergonomic studies, and a maximum output force and momentum F_M expected on P is defined. To choose the actuators we need to know the value of the maximum tension in (W_{pr}) for $||\mathbf{F}_P|| \leq F_M$.A first approach would be, for each point of (W_{pr}) and for every possible configurations of the output force, to compute the tension in each cable and then to get the maximum tension. Even if the calculation time is quite small, it would take hours to get the solution. Moreover, it is not possible to warrant that the worst case is obtained as all the possible configurations are not explored. That why we propose a new approach. To be able to find the simplest analytical solutions we have assumed in this section that t_{min} is small enough to be considered as null.

3.2. Calculation of the maximum tension

Let's write:

- $G = [g_{ij}],\ i \in [1, n], j \in [1, m]$
- $\mathbf{F}_P = [\mathbf{F}_{P1}, \dots, \mathbf{F}_{Pn}]^t$

Let's calculate $|A(F)^{S_k}_{c_r}| = \Delta^{S_k}_{c_r}(\mathbf{F}_P)$ by developing it following the c_r column

$$\Delta^{S_k}_{c_r}(\mathbf{F}_P) = \begin{vmatrix} g_{1c_1} & \cdots & g_{1c_{r-1}} & \mathbf{F}_{P1} & g_{1c_{r+1}} & \cdots & g_{1c_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{nc_1} & \cdots & g_{nc_{r-1}} & \mathbf{F}_{Pn} & g_{nc_{r+1}} & \cdots & g_{nc_n} \end{vmatrix}$$

$$= F_{P1}(-1)^{i+c_r} \underbrace{\begin{vmatrix} g_{2c_1} & \cdots & g_{2c_{r-1}} & g_{2c_{r+1}} & \cdots & \mathbf{g}_{2c_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{nc_1} & \cdots & g_{nc_{r-1}} & g_{nc_{r+1}} & \cdots & g_{nc_n} \end{vmatrix}}_{\Delta^{S_k}_{n-1}(1,c_r)}$$

$$+ \dots + (-1)^{n+c_r}\Delta^{S_k}_{n-1}(n, c_r) = \sum_{i=1}^{n} \mathbf{F}_{Pi}(-1)^{i+c_r}\Delta^{S_k}_{n-1}(i, c_r) \tag{8}$$

Thus :

$$\begin{aligned} t^{S_k}_{c_r} &= \frac{\mathbf{F}^*_P \vec{\Delta}^{S_k}_{n-1}(c_r)}{\Delta^{S_k}_n} \\ \vec{\Delta}^{S_k}_{n-1}(c_r) &= [(-1)^{1+c_r}\Delta^{S_k}_{n-1}(1, c_r), \dots, (-1)^{n+c_r}\Delta^{S_k}_{n-1}(n, c_r)]^t \end{aligned} \tag{9}$$

The maximum of $t^{S_k}_{c_r}$ will be obtained by finding the upper bound of the numerator:

$$t^{S_k}_{c_r} \leq \frac{||\mathbf{F}^*_P||||\vec{\Delta}^{S_k}_{n-1}(c_r)||}{\Delta^{S_k}_n} \tag{10}$$

For $t_{min} \approx 0$, we will have $||\mathbf{F}^*_P|| \approx ||\mathbf{F}_P|| = F_M$, so we can write :

$$t^{S_k}_{c_r} \leq \frac{F_M||\vec{\Delta}^{S_k}_{n-1}(c_r)||}{\Delta^{S_k}_n} \tag{11}$$

To obtain the maximum on the cable i, we need to test all the possible sets of cables S_k including cable i. Let's $S^r_k(i)$ be those cables sets, there are $C^{n-1}_{m-1} = \frac{(m-1)!}{(n-1)!(m-n)!}$ of them, so for $r \in [1, C^{n-1}_{m-1}]$:

$$t_{i\,max} \approx F_M \max_{r \in [1, C^{n-1}_{m-1}]} \frac{\sqrt{\sum_{j=1}^{n}(\Delta^{S^r_k(i)}_{n-1}(j, f^{-1}_k(i)))^2}}{\Delta^{S^r_k(i)}_n} \tag{12}$$

We finally have an expression of the maximum of the tension in each cable which is only linked to the norm of $\mathbf{F}_P$ and the elements of the matrix G .

3.2.1. 2 d.o.f, 3 cables planar interface

In this example we have p=1 et m=3. Using the second approach for the calculation of the tension in the cables gives:

$$\begin{aligned} S_1 &= \{1, 2\},\ c_1 = 1, c_2 = 2, c_3 = 3, c_4 = 4,\ G^{S_1}_{red} = [\vec{u_1}, \vec{u_2}] \\ S_2 &= \{2, 3\},\ c_1 = 2, c_2 = 3, c_3 = 1, c_4 = 4,\ G^{S_2}_{red} = [\vec{u_2}, \vec{u_3}] \\ S_3 &= \{1, 3\},\ c_1 = 1, c_2 = 3, c_3 = 2, c_4 = 4,\ G^{S_3}_{red} = [\vec{u_1}, \vec{u_3}] \end{aligned}$$

The expression of $t^{S_k}_i$ can be simplified as follow:

$$t^{S_k}_i = \left.\frac{|\vec{F_{out}}, \vec{u_j}|}{|\vec{u_i}, \vec{u_j}|}\right|_{j \in S_k, j \neq i} \tag{13}$$

So the maximum of t_i will be

$$\begin{aligned} t^{max}_i &= \max_{j=\{[1,3]-i\}} \left\{ \frac{|\vec{u_i}, \vec{F_{out}}|}{|\vec{u_i}, \vec{u_j}|} \right\} \\ &= \max_{j=\{[1,3]-i\}} \left\{ \frac{\sin(\vec{u_i}, \vec{F_{out}})||\vec{F_{out}}||}{\sin\theta_{ij}} \right\} \end{aligned}$$

Let's write $\vec{F_{out}} = F_M \vec{u_F}$

$$\begin{aligned} \Rightarrow t^{max}_i &= \max_{j=\{[1,3]-i\}} \left\{ \frac{|\sin(\vec{u_i}, \vec{u_F})|F_M}{\sin\theta_{ij}} \right\} \\ &= \max_{j=\{[1,3]-i\}} \left\{ \frac{F_M}{\sin\theta_{ij}} \right\} \end{aligned} \tag{14}$$

For example :

$$t^{max}_1 = \max\left\{ \frac{F_M}{\sin\theta_{12}}, \frac{F_M}{\sin\theta_{13}} \right\} \tag{15}$$

3.2.2. 2 d.o.f, 4 cables planar interface

In this example we have p=2 et m=4.
Using the second approach for the calculation of the tension in the cables gives:

$$\begin{aligned} S_1 &= \{1, 2\},\ c_1 = 1, c_2 = 2, c_3 = 3, c_4 = 4,\ G^{S_1}_{red} = [\vec{u_1}, \vec{u_2}] \\ S_2 &= \{2, 3\},\ c_1 = 2, c_2 = 3, c_3 = 1, c_4 = 4,\ G^{S_2}_{red} = [\vec{u_2}, \vec{u_3}] \\ S_3 &= \{3, 4\},\ c_1 = 3, c_2 = 4, c_3 = 1, c_4 = 2,\ G^{S_3}_{red} = [\vec{u_3}, \vec{u_4}] \\ S_4 &= \{4, 1\},\ c_1 = 4, c_2 = 1, c_3 = 2, c_4 = 3,\ G^{S_4}_{red} = [\vec{u_4}, \vec{u_1}] \\ S_5 &= \{1, 3\},\ c_1 = 1, c_2 = 3, c_3 = 2, c_4 = 4,\ G^{S_5}_{red} = [\vec{u_1}, \vec{u_3}] \\ S_6 &= \{2, 4\},\ c_1 = 2, c_2 = 4, c_3 = 2, c_4 = 3,\ G^{S_6}_{red} = [\vec{u_2}, \vec{u_4}] \end{aligned}$$

The expression of $t^{S_k}_i$ is then:

$$t^{S_k}_i = \left.\frac{|\vec{F_{out}}, \vec{u_j}|}{|\vec{u_i}, \vec{u_j}|}\right|_{j \in S_k, j \neq i} \tag{16}$$

So the maximum of t_i will be

$$\Rightarrow t^{max}_i = \max_{j=\{[1,4]-i\}} \left\{ \frac{F_M}{\sin\theta_{ij}} \right\} \tag{17}$$

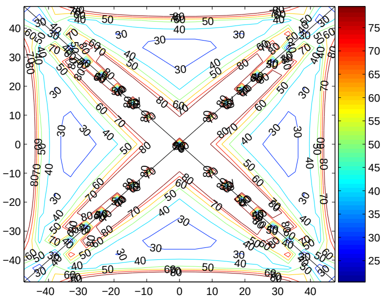

Figure 4: Drawing of the lines of equal first maximum tension on the four cables in the theoretic workspace for an output force of 10 N

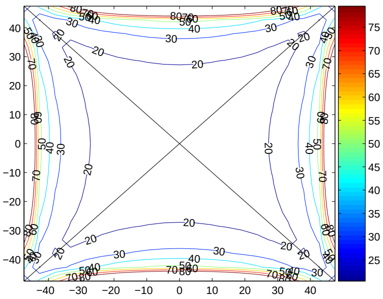

Figure 5: Drawing of the lines of equal second maximum tension on the four cables in the theoretic workspace for an output force of 10 N

In this example, because of the redundancy, the approach has to be modified. Indeed if we use relation 17, we will obtained the following lines of equal maximum tension (fig 4).

This is happening because, with this solution, the maximum is obtained when two cables are aligned (i.e :$|\sin\theta_{ij}| = 0$), but because of redundancy, we choose, to calculate the tension in the cables, the minimum of the positive solutions; so this configuration is always rejected. In order to be able to draw the correct repartition of the tension in the workspace we have to choose the second maximum tension, which is smaller. We then obtain the correct drawing of the lines of equal maximum tension on the four cables

3.2.3. 3 d.o.f, 4 cables spatial interface example

The results previously demonstrated can be extended to the spatial case. In this example we have p=1 et m=4. Using the second approach for the calculation of the tension in the cables gives:

$$S_1 = \{1,2,3\},\ c_1 = 1, c_2 = 2, c_3 = 3, c_4 = 4,\ G_{red}^{S_1} = [\vec{u_1}, \vec{u_2}, \vec{u_3}]$$
$$S_2 = \{2,3,4\},\ c_1 = 2, c_2 = 3, c_3 = 1, c_4 = 4,\ G_{red}^{S_2} = [\vec{u_2}, \vec{u_3}, \vec{u_4}]$$
$$S_3 = \{3,4,1\},\ c_1 = 3, c_2 = 4, c_3 = 1, c_4 = 2,\ G_{red}^{S_3} = [\vec{u_3}, \vec{u_4}, \vec{u_1}]$$
$$S_4 = \{4,1,2\},\ c_1 = 4, c_2 = 1, c_3 = 2, c_4 = 3,\ G_{red}^{S_4} = [\vec{u_4}, \vec{u_1}, \vec{u_2}]$$

The expression of $t_i^{S_k}$ obtained with this approach can be simplified as follow:

$$t_i^{S_k} = \left.\frac{|\vec{F_{out}}, \vec{u_j}, \vec{u_k}|}{|\vec{u_i}, \vec{u_j}, \vec{u_k}|}\right|_{\{j,k\}\in {S_k}^2,\ j\neq k\neq i}$$

So the maximum of t_i for a given maximum output force will be

$$t_i^{max} = \max_{\{j,k\}\in\{[1,4]-i\}^2,\ j\neq,k} \left\{ \frac{|\sin(\vec{u_j}, \vec{u_k})| F_M}{|\vec{u_i}, \vec{u_j}, \vec{u_k}|} \right\} \tag{18}$$

For example:

$$t_1^{max} = \max\left\{ \frac{|\sin(\vec{u_2}, \vec{u_3})| F_M}{|\vec{u_1}, \vec{u_2}, \vec{u_3}|}, \frac{|\sin(\vec{u_3}, \vec{u_4})| F_M}{|\vec{u_1}, \vec{u_3}, \vec{u_4}|}, \frac{|\sin(\vec{u_2}, \vec{u_4})| F_M}{|\vec{u_1}, \vec{u_2}, \vec{u_4}|} \right\} \tag{19}$$

4. Calculation of the Tension Capable Workspace

This workspace can be defined as :

$$P \in W_{t_{max}} \Leftrightarrow \left\{ \begin{array}{c} ||\mathbf{F}_P|| < F_M \\ \mathbf{F}_P = G\mathbf{T},\ t_{min} < t_i < t_{max} \end{array} \right\} \tag{20}$$

The relations between F_M and t_{max}, previously found gives the equation of the boundaries of $W_{t_{max}}$

4.1. 2 d.o.f, 3 cables interface

Knowing F_M and t_{max} we can deduce from (14) the maximum possible angle between two cables :

$$\theta_{max} = \pi - \sin^{-1}\left(\frac{F_M}{t_{max}}\right) \tag{21}$$

This relation is the equation of three circles which are going trough two of the three points A_i and a point M being the third point of a triangle A_iA_jM in witch $(\vec{MA_i}, \vec{MA_j}) = \theta_m$. So the largest circular workspace is given by the maximum circle tangent to these previous ones. A geometrical study gives:

$$Rg_{T\,max} = \frac{r(1 - \sqrt{3}\tan(\frac{\pi - \theta_{max}}{2}))}{2} \tag{22}$$

The theoretical results have been compared to those obtained by calculation of the tension in the cables across 1620 points of the theoretical workspace, for 30^2 different orientation of a 10 N normed force. As shown on fig 8 the theoretical boundaries (in blue) perfectly match with the simulated values of the tension in the cables: the yellow points, which are the points where the tension in the cables is less than the chosen value of 10 N, are inside the calculated boundaries. As expected the maximum circular workspace (in green) is tangent to the boundaries.

4.2. 3 d.o.f, 4 cables spatial interface

From (19) we can deduce the relation between three angles, F_M and t_{max}:

$$C_{max} = \frac{F_M}{t_{max}} = \max_{\substack{i=1,4 \\ j=1,4 \\ k=1,4 \\ i\neq j\neq k}} \left\{ \frac{|(\vec{u_i}, \vec{u_j}, \vec{u_k})|}{||\vec{u_i} \wedge \vec{u_j}||} \right\} \tag{23}$$

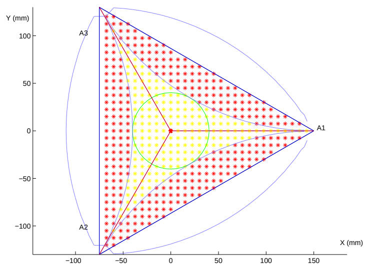

Figure 6: The simulated Tension capable workspace (yellow points) and the calculated boundaries (in blue)

The equations of boundaries of $W_{t_{max}}$ are defined by :

$$\frac{|(\vec{u_i}, \vec{u_j}, \vec{u_k})|}{||\vec{u_i} \wedge \vec{u_j}||} = \frac{(\vec{u_i} \wedge \vec{u_j}) \cdot \vec{u_k}}{||\vec{u_i} \wedge \vec{u_j}||} = \cos(\vec{n_{ij}}, \vec{u_k}) = C_{max} \qquad (24)$$

With $\vec{n_{ij}}$ being the unitary vector perpendicular to the plane formed by the vectors $\vec{u}_i, \vec{u}_j$

As there is no simple analytical definition of the surface generated by the previous equation, we can compute the definition in order to be able to draw it (fig 7).

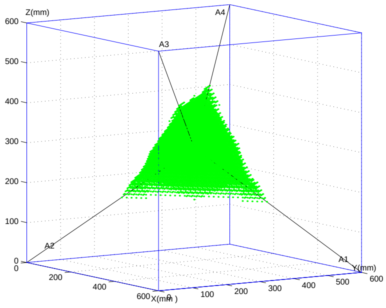

Figure 7: The simulated Tension capable workspace (green), $t_{max} = 20N$, $F_N = 10N$, 9.10^3 pts of simulation

4.3. 2 d.o.f, 4 cables interface

The previous relation giving the maximum value for θ_{max} can still be used. It will give the boundaries of $W_{t_{max}}$ of this interface. The boundaries are then four circles which are going trough three of the four points A_i and a point M being the third point of a triangle A_iA_jM in witch $(\vec{MA_i}, \vec{MA_j}) = \theta_m$. This can be easily seen on the following figure (fig. 8) which is the result of the computation of the maximum tension calculation in 50*50 points of W_{th} for a maximum output of 5 N and a maximum tension of 10 N.

5. Conclusion

In this paper, we have presented three kinds of Completely Constrained Punctual Cable Robot and explained why and how they can be used as haptic interfaces. We have developed two approaches to calculate the maximum tension in the cables for any given point of a chosen workspace and for an output force of any direction, for any type of cable driven haptic interface. Then we have deduced what we called the Tension Capable Workspace, and we have described the way to choose the workspace in order to be able to control the interface in any point for a given set of motors. Future work will focus on finding an analytic description of the Tension Capable Workspace in the spatial case an to extend this method to design a new 3 d.o.f, 4 cables light haptic interface. We will also focus on extend this method to more complex mechanisms.

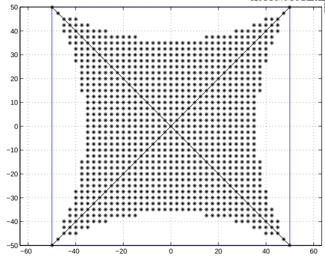

Figure 8: The simulated Tension capable workspace

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